Marks : 40	SYJC March' 19 Subject : Maths – I Derivatives & Integration	Duration : 1.5 Hours. Set – A SOLUTION

Q.1. Attempt any Two. (2 Marks each) **(04)**

1. The total cost function is given by as C = 1200x – 20x^2 + 338.

$\therefore \frac{dC}{dx} = \frac{d}{dx}\left(1200x - 20x^2 + 338\right)$

$\therefore \frac{dC}{dx} = 1200 \text{ x } 1 - 20 \text{ x } 2x + 0$

$\therefore \frac{dC}{dx} = 1200 - 40x$

If the total cost is decreasing, then $\frac{dC}{dx} < 0$

$\therefore$ 1200 – 40x < 0

$\therefore$ 1200 < 40x

$\therefore$ x > 30

Hence, the total cost is decreasing for x > 30.

2. Let R be the total revenue

Then R = p.x = (280 - x)x

$\therefore$ R = 280x – x^2

$\therefore \frac{dR}{dx} = \frac{d}{dx}\left(280x - x^2\right)$

$\therefore \frac{dR}{dx} = 280 \text{ x } 1 - 2x = 280 - 2x$

Revenue is increasing, if $\frac{dR}{dx} > 0$

i.e., 280 – 2x > 0

i.e., 280 > 2x

i.e., x < 140

Hence, the revenue is increasing, if x < 140.

3. The total cost function is given by as C = 225 – 36x + x^2

$\therefore \frac{dC}{dx} = \frac{d}{dx}\left(225 - 36x + x^2\right)$

$\therefore \frac{dC}{dx} = 0 - 36 \text{ x } 1 + 2x$

$\therefore \frac{dC}{dx} = -36 + 2x$

If the total cost is decreasing, then $\frac{dC}{dx} < 0$

i.e., – 36 + 2x < 0

i.e., 2x < 36

i.e., x < 18

Hence, the revenue is increasing, if x < 18.

Q.2. Attempt any Four. (3 Marks each) **(12)**

1. Show that the following functions are always increasing:

$f(x) = x^3 - 6x^2 + 12x + 10$

$\therefore\ f'(x) = \frac{d}{dx}\left(x^3 - 6x^2 + 12x + 10\right)$

$\therefore\ f'(x) = 3x^2 - 6 \times 2x + 12 \times 1 + 0$

$\therefore\ f'(x) = 3x^2 - 12x + 12$

$\therefore\ f'(x) = 3\left(x^2 - 4x + 4\right) = 3\left(x - 2\right)^2 > 0$ for all $x \in R$, $x \neq 2$

$\therefore$ f'(x) > 0 for all $x \in R - \{2\}$

$\therefore$ f is increasing for all $x \in R - \{2\}$.

2. The total cost function is given by as $C = 100 + 600x - 3x^2$

$\therefore\ \frac{dC}{dx} = \frac{d}{dx}\left(100 + 600x - 3x^2\right)$

$\therefore\ \frac{dC}{dx} = 0 + 600 \times 1 - 3 \times 2x$

$\therefore\ \frac{dC}{dx} = 600 - 6x$

If the total cost is decreasing, then $\frac{dC}{dx} < 0$

$\therefore$ 600 – 6x < 0

$\therefore$ 600 < 6x

$\therefore$ x > 100

Hence, the revenue is increasing, if x > 100.

3. Total cost C = Labour cost + processing cost

$\therefore$ $C = 150 - 54x + x^2$

$\therefore\ \frac{dC}{dx} = \frac{d}{dx}\left(150 - 54x + x^2\right)$

$\therefore\ \frac{dC}{dx} = 0 - 54 \times 1 + 2x = -54 + 2x$

If the total cost is decreasing, then $\frac{dC}{dx} < 0$

i.e., – 54 + 2x < 0

i.e., 2x < 54

i.e., x < 27

Hence, the revenue is increasing, if x < 27.

4. The total revenue R is given by

R = p.x = (120 - x)x

$\therefore$ $R = 120x - x^2$

$\therefore\ \frac{dR}{dx} = \frac{d}{dx}\left(120x - x^2\right)$

$\therefore\ \frac{dR}{dx} = 120 \times 1 - 2x = 120 - 2x$

If the revenue is increasing, if $\frac{dR}{dx} > 0$

$\therefore$ 120 – 2x > 0

$\therefore$ 120 > 2x

$\therefore$ x < 60

Hence, the revenue is increasing when x < 60.

5. $E_c = (0.0003)I^2 + (0.075)I$

$$MPC = \frac{dE_c}{dI} = \frac{d}{dI}\left[(0.0003)I^2 + (0.075)I\right]$$

$$MPC = (0.0003)(2I) + (0.075)(1)$$

$$MPC = (0.0006)I + 0.075$$

When I = 1000, then

$$MPC = (0.0006)(1000) + 0.075$$

$$MPC = 0.6 + 0.075 = 0.675$$

$\therefore$ MPC + MPS = 1

$\therefore$ 0.675 + MPS = 1

$\therefore$ MPS = 1 – 0.675 = 0.325

Now, $APC = \frac{E_c}{I} = \frac{(0.0003)I^2 + (0.075)I}{I}$

$$APC = \frac{E_c}{I} = (0.0003)I + (0.075)$$

When I = 1000, then

$$APC = (0.0003)(1000) + 0.075$$

$$APC = 0.3 + 0.075 = 0.375$$

$\therefore$ APC + APS = 1

$\therefore$ 0.375 + APS = 1

$\therefore$ APS = 1 – 0.375 = 0.625

Hence, MPC = 0.675, MPS = 0.325

APC = 0.375, APS = 0.625.

6. Let R be the total revenue

Then R = p.x. = (280 - x)x

$\therefore$ $R = 280x – x^2$

Let C be the total cost

Then C = processing cost + standing charges

$\therefore$ C = 2x + 320

Profit $\pi = R – C = (280x – x^2) – (2x + 320)$

Profit $\pi = R – C = 280x – x^2 – 2x – 320$

Profit $\pi = 280x – x^2 – 2x – 320$

$$\frac{d\pi}{dx} = \frac{d}{dx}\left(278x - x^2 - 320\right)$$

$$\frac{d\pi}{dx} = 278 \times 1 - 2x - 0 = 278 - 2x$$

Profit is increasing, if $\frac{d\pi}{dx} > 0$

$\therefore$ 278 – 2x > 0

$\therefore$ 278 > 2x

$\therefore$ x < 139

Hence, the revenue is increasing when x < 139.

7. The demand function is $D = \frac{36}{p+1}$

$\therefore \frac{dD}{dp} = \frac{d}{dp}\left(\frac{36}{p+1}\right) = 36\frac{d}{p}(p+1)^{-1}$

$\therefore \frac{dD}{dp} = (36)(-1)(p+1)^{-2}.\frac{d}{dp}(p+1)$

$\therefore \frac{dD}{dp} = \frac{-36}{(p+1)^2}.(1+0) = \frac{-36}{(p+1)^2}$

Elasticity of demand is given by

$\eta = -\frac{p}{D}.\frac{dD}{dp}$

$\eta = \frac{-p}{\left(\frac{36}{p+1}\right)} x \frac{-36}{(p+1)^2} = \frac{p}{p+1}$

When p = 2, $\eta = \frac{2}{2+1} = \frac{2}{3}$

Since, 0 < η < 1, the demand is relatively inelastic.

Q.3. Attempt any One. (4 Marks each) (04)

1. The total cost C is given by

C= processing cost + packing and dispatching cost

= $\left(\frac{2x^3}{3} - 48x^2\right) + (1289x + 3750)$

$\therefore C = \frac{2x^3}{3} - 48x^2 + 1289x + 3750$

$\therefore$ marginal cost = $C_m = \frac{dC}{dx}$

$= \frac{d}{dx}\left(\frac{2x^3}{3} - 48x^2 + 1289x + 3750\right)$

$= \frac{2}{3} \times 3x^2 - 48 \times 2x + 1289 \times 1 + 0$

$\therefore C_m = 2x^2 - 96x + 1289$

$\therefore \frac{dC_m}{dx} = \frac{d}{dx}(2x^2 - 96x + 1289)$

$\therefore \frac{dC_m}{dx} = 2 \times 2x - 96 \times 1 + 0$

$\therefore \frac{dC_m}{dx} = 4x - 96$

And $\frac{d^2C_m}{dx^2} = \frac{d}{dx}(4x - 96)$

$\frac{dC_m}{dx} = 0$ gives 4x – 96 = 0

$\therefore$ x = 24

And $\left(\frac{d^2C_m}{dx^2}\right)_{at\ x=24} = 4 > 0$

$\therefore$ by the second derivatives test.

C_m is minimum when x = 24 and

the marginal cost at x = 24 is $(2x^2 - 96x + 1289)_{at\ x = 24} = 2(24)^2 - 96(24) + 1289$

$= 1152 - 2304 + 1289$

$= 137$

Hence, the marginal cost is minimum, when the number of bags manufactured is 24 and the marginal cost is 137.

2. The total cost is given as

$C = 11Q - 7Q^2 + Q^3$

$\therefore \frac{dC}{dQ} = \frac{d}{dQ}(11Q - 7Q^2 + Q^3)$

$\therefore \frac{dC}{dQ} = 11 \times 1 - 7 \times 2Q + 3Q^2$

$\therefore \frac{dC}{dQ} = 11 - 14Q + 3Q^2$

And $\frac{d^2C}{dQ^2} = \frac{d}{dQ}(11 - 14Q + 3Q^2)$

$\frac{d^2C}{dQ^2} = 0 - 14 \times 1 + 3 \times 2Q$

$\frac{d^2C}{dQ^2} = -14 + 6Q$

$\frac{dC}{dQ} = 0$

$11 - 14Q + 3Q^2 = 0$

$3Q^2 - 11Q - 3Q + 11 = 0$

$Q(3Q - 11) - 1(3Q - 11) = 0$

$(3Q - 11)(Q - 1) = 0$

$\therefore Q = \frac{11}{3}$ or $Q = 1$

$\left(\frac{d^2C}{dQ^3}\right)_{at\ Q=1} = -14 + 6 \times 1 = -8 < 0$

$\therefore$ by the second derivative test, C is maximum, when Q = 1

Also, $\left(\frac{d^2C}{dQ^3}\right)_{at\,Q=\frac{11}{3}} = -14+6\,x\,\frac{11}{3} = 8 > 0$

$\therefore$ by the second derivative test.

C is minimum, when $Q = \frac{11}{3}$

Now, average cost = $C_A = \frac{C}{Q}$

= $\frac{11Q - 7Q^2 + Q^3}{Q}$

$\therefore$ C_A = 11 – 7Q + Q^2

$\therefore$ $\frac{dC_A}{dQ} = \frac{d}{dQ}\left(11 - 7Q + Q^2\right)$

$\frac{dC_A}{dQ} = 0 - 7\times1 + 2Q = -7 + 2Q$

And $\frac{d^2C_A}{dQ^2} = \frac{d}{dQ}\left(-7+2Q\right)$

$\frac{d^2C_A}{dQ^2} = -0 + 2\times1 = 2$

$\frac{dC_A}{dQ} = 0$ gives – 7 + 2Q = 0

$\therefore$ $Q = \frac{7}{2}$

and $\left(\frac{d^2C_A}{dQ^2}\right)_{at\,Q=\frac{7}{2}} = 2 > 0$

$\therefore$ by the second derivative test,

C_A is minimum when $Q = \frac{7}{2}$

Hence, the total cost is minimum, when $Q = \frac{11}{3}$ and the average cost is minimum, when $Q = \frac{7}{2}$.

3. The demand function is D = 50 – 3p – p^2

$\therefore \frac{dD}{dp} = \frac{d}{dp}\left(50 - 3p - p^2\right)$

$\therefore$ $\frac{dD}{dp} = 0 - 3\times1 - 2p = -3 - 2p$

Elasticity of demand is given by

$\eta = -\frac{p}{D}.\frac{dD}{dp}$

$\eta = \frac{-p}{\left(50 - 3p - p^2\right)}\,x\left(-3 - 2p\right)$

$$\eta = \frac{p(3+2p)}{50-3p-p^2}$$

(i) When p = 5, then

$$\eta = \frac{5(3+2\times5)}{50-3(5)-(5)^2} = \frac{5\times13}{50-15-25} = \frac{65}{10} = 6.5$$

(ii) When p = 2, then

$$\eta = \frac{2(3+2\times2)}{50-3(2)-(2)^2} = \frac{2\times7}{50-6-4} = \frac{14}{40} = \frac{7}{20}$$

Q.4. Attempt any Two : (2 Marks each) **(04)**

1. $\int (3x + 4x)^3\, dx$

$= \dfrac{(3x+4)^4}{4\times3} + c$

$= \dfrac{(3x+4)^4}{12} + c$

2. $\displaystyle\int \frac{1}{1+\cos x}dx = \int \frac{1}{2\cos^2\left(\frac{x}{2}\right)}dx$

$= \dfrac{1}{2}\displaystyle\int \sec^2\left(\frac{x}{2}\right)dx = \frac{1}{2}\, x\, \frac{\tan(x/2)}{(1/2)} + c$

$= \tan \dfrac{x}{2} + c$

Alternative Method:

$\displaystyle\int \frac{1}{1+\cos x}dx = \int \frac{1}{1+\cos x}\ x\ \frac{1-\cos x}{1-\cos x}\ dx$

$= \displaystyle\int \frac{1-\cos x}{1-\cos^2 x}dx = \int \frac{1-\cos x}{\sin^2 x}\ dx$

$= \displaystyle\int \left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right)dx$

$= \displaystyle\int \left(\text{cosec}^2 x - \frac{1}{\sin x}\ x\ \frac{\cos x}{\sin x}\right)dx$

$= \displaystyle\int \text{cosec}^2 x\, dx - \int \text{cosec}\, x \times \cot x\, dx$

$= -\cot x - (-\text{cosec}\, x) + c$

$= \text{cosec}\, x - \cot x + c.$

3. Let $I = \displaystyle\int e^x \sec x (1 + \tan x) dx$

$= \displaystyle\int e^x (\sec x + \sec x \times \tan x) dx$

Put $f(x) = \sec x$

$\therefore \quad f'(x) = \sec x \cdot \tan x$

$\therefore \quad I = \displaystyle\int e^x \left[f(x) + f'(x)\right] dx$

$= e^x \times f(x) + c = e^x \times \sec x + c$

Q.5. Attempt any Four : (3 Marks each). **(12)**

1. $\int \frac{9x^4}{\sqrt{x}}\,dx + 5\int \frac{x^2}{\sqrt{x}}\,dx + \int \frac{1}{\sqrt{x}}\,dx$

$= 9\int x^{\frac{7}{2}}\,dx + 5\int x^{\frac{3}{2}} + \int x^{\frac{-1}{2}}\,dx$

$= 9\frac{x^{\frac{9}{2}}}{\frac{9}{2}} + 5\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2x^{\frac{9}{2}} + 2x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + c$

2. Let $I = \frac{(\sin x)}{(1-\sin x)}dx$

$I = \int \frac{\sin x(1+\sin x)}{(1-\sin x)(1+\sin x)}dx$

$I = \int \frac{(\sin x + \sin^2 x)}{(1-\sin^2 x)}dx$

$I = \int \left[\frac{\sin x + \sin^2 x}{\cos^2 x}\right]dx$

$I = \int \left[\frac{\sin x}{\cos x}\frac{1}{\cos x} + \frac{\sin^2 x}{\cos^2 x}\right]dx$

$I = \int \left[(\sec x)(\tan x) + \tan^2 x\right]dx$

$I = \int \left[\sec x \tan x + \sec^2 x - 1\right]dx$

$I = \int (\sec x \tan x)\,dx + \int (\sec^2 x)\,dx - \int (1)\,dx \quad I = \sec x + \tan x - x + c$

3. $I = \int \frac{dx}{x^2 + x + \frac{1}{4} + 1 - \frac{1}{4}}$

$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$

$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{1}{\sqrt{3/2}} \tan^{-1}\left(\frac{x + 1/2}{\sqrt{3/2}}\right) + c$

$I = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

4. $$\int \frac{dx}{\sqrt{\left(x^2+4x+4\right)+\left(5-4\right)}}$$

$$= \int \frac{dx}{\sqrt{\left(x+2\right)^2+\left(1\right)^2}}$$

$$= \log\left|\left(x+2\right)+\sqrt{\left(1\right)^2+\left(x+2\right)^2}\right| + c$$

$$= \log\left|\left(x+2\right)+\sqrt{x^2+4x+5}\right| + c$$

5. Let $$I = \int_0^{\frac{\pi}{2}} \left(\cos^2 x\right) dx$$

$$I = \int_0^{\frac{\pi}{2}} \left[\frac{1+\cos 2x}{2}\right] dx$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1\right) dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\cos 2x\right) dx$$

$$I = \frac{1}{2}\left[x\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$I = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) + \frac{1}{4}\left(\sin\pi - \sin 0\right)$$

$$I = \frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\left(0-0\right)$$

$$I = \frac{\pi}{4}$$

6. $$\int_1^2 \frac{dx}{x^2+6x+5}$$

$$= \int_1^2 \frac{dx}{\left(x^2+6x+9\right)-4}$$

$$= \int_1^2 \frac{1}{\left(x+3\right)^2-\left(2\right)^2} dx$$

$$= \frac{1}{2\left(2\right)}\left[\log\left|\frac{x+3-2}{x+3+2}\right|\right]_1^2$$

$$= \frac{1}{4}\left[\log\left|\frac{x+1}{x+5}\right|\right]_1^2$$

$$= \frac{1}{4}\left[\log\frac{3}{7} - \log\frac{2}{6}\right]$$

$$= \frac{1}{4}\log\left(\frac{3}{7} \times \frac{6}{2}\right)$$

$$= \frac{1}{4}\log\left(\frac{9}{7}\right)$$

Q.6. Attempt any One : (4 Marks each). **(04)**

1. $\int 3\sqrt{x^2 - \frac{6x}{9} + \frac{4}{9}}\ dx$

$= 3\int \sqrt{x^2 - \frac{6}{9}x + \frac{1}{9} + \frac{4}{9} - \frac{1}{9}}$

$= 3\int \sqrt{\left(x - \frac{1}{3}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2}$

$= 3\left[\frac{\left(x - \frac{1}{3}\right)}{2}\sqrt{x^2 - \frac{6}{9}x + \frac{4}{9}} - \frac{1}{6}\log\left|\left(x - \frac{1}{3}\right) + \sqrt{x^2 - \frac{6x}{9} + \frac{4}{9}}\right| + c\right.$

2. Let $I = \int \frac{x+4}{\left(x^2 + 6x + 10\right)} dx$

$I = \frac{1}{2}\int \frac{2x+8}{\left(x^2+6x+10\right)} dx$

$= \frac{1}{2}\int \left[\frac{(2x+6)+2}{\left(x^2+6x+10\right)}\right] dx$

$= \frac{1}{2}\int \frac{2x+6}{x^2+6x+10} dx + 1\int \frac{1}{x^2+6x+10} dx \left[\frac{b}{a} = \frac{6}{1} \therefore \frac{b^2}{4a^2} = \frac{36}{4} = 9 \therefore \frac{b}{2a} = 3\right]$

$= \frac{1}{2}\int \frac{2x+6}{x^2+6x+10} + \int \frac{dx}{(x+3)^2 + 1^2}$

$= \frac{1}{2}\log\left|x^2+6x+10\right| + \frac{1}{1}\tan^{-1}(x+3) + c$

$= \frac{1}{2}\log\left|x^2+6x+10\right| + \tan^{-1}(x+3) + c$

3. $\int_0^1 \frac{x^2+3x+2}{\sqrt{x}} dx$

$= \int_0^1 \frac{x^2}{\sqrt{x}} dx + \int_0^1 \frac{3x}{\sqrt{x}} dx + 2\int_0^1 \frac{1}{\sqrt{x}} dx$

$= \int_0^1 x^{3/2} dx + 3\int_0^1 x^{1/2} dx + 2\int_0^1 x^{-1/2} dx$

$= \left[\frac{x^{5/2}}{5/2}\right]_0^1 + 3\left[\frac{x^{3/2}}{3/2}\right]_0^1 + 2\left[\frac{x^{1/2}}{1/2}\right]_0^1$

$= \frac{1}{5/2} + 3\left[\frac{1}{3/2}\right] + 2\left[\frac{1}{1/2}\right]$

$= \frac{2}{5} + 2 + 4$

$= \frac{2}{5} + 6$

$= \frac{32}{5}$